A Twist on the Monty Hall Problem
- Summary of ALLSTAT Discussion-

February 2009

Nigel Marriott
Statistical Consultant

Email: **nigel@marriott-stats.com** Website: **www.marriott-stats.com**

1. Introduction

This report summarises the responses I received to a query I put on ALLSTAT about the classic Monty Hall problem. The background to this question is that I often use the Monty Hall game show as an icebreaker for my training courses as it is a classic demonstration of how normal human intuition fails when statistical thinking is required. I then like to show an example of how the Monty Hall solution can be applied in real-life. My current example is a competitive tender which is explained in a little later on in this note but in effect what I am seeking from people is whether there is a better example. In Colin Bruce's book "Conned Again Watson!", he creates a Sherlock Holmes story involving 3 graves, one of which contains the supposed victim of a murder as his equivalent Monty Hall problem. Colin's book is an excellent one and I use it a lot in my courses but I do find his corollary somewhat contrived and my competitive tender is the best I can do at the moment.

In this report, I will first list the responses to my first query. These showed that I had probably not fully specified the rules of the game so I issued a second query which elicited more responses. I have copied everybody's responses to both queries in this report because some of them were very long and I felt it was fairer to include them in full. Email addresses are given so that you can respond direct to the person concerned if you wish.

Finally can I thank everyone for their time in responding to my queries? Your responses gave me much food for thought and shows that the Monty Hall problem can still simulate debate among statisticians!

2a. My First Query to ALLSTAT

“I am hoping you can help me with a twist on the classic Monty Hall problem. For those of you who don’t know, this is the game show where there are 3 doors that a contestant has to choose from, one of which contains the prize. After choosing one, the game show host opens one of other two doors that the host knows is not a winning door. The contestant then has the option to STICK with his original choice or CHANGE his door to the other unopened door. What surprises people is that the probability of winning with the Change strategy is 2/3 not 50%.

A related problem is something that I have seen described as a the prisoner’s dilemma. In this scenario, 3 prisoners are waiting to be executed but in a spirit of mercy, the king will pardon two of them and only execute one of them. The prison warden visits one of the prisoners and tells him that one of the other prisoners is going to be pardoned. At first sight this is exactly the same as the Monty Hall problem in that being a prisoner is the first door chosen and the warden is the game show host. But the difference is that the prisoner is playing a STICK strategy since he can’t change places with other prisoners.

My question concerns the King. Is there a difference between a situation where the king says to the warden “I’ve made my mind up. All I will tell you for now is that prisoner B will definitely be pardoned” and the warden goes to A (his friend) to tell him this and the situation where the king says “I’ve haven’t decided yet. I definitely going to pardon B but can’t decide which one of A and C to execute.” It seems to me that the former is a parallel of the Monty Hall show in that the king (via the warden) is acting as the game show host who knows what the actual outcome is whereas in the latter, the king doesn’t yet know what the outcome is going to be. Yet prisoner A will receive the same information regardless.”

2b. Responses to First Query

Take particular note of Rebecca Pillinger's reply, surely the longest ever submitted to an ALLSTAT query!

From Eryl Bassett, E.E.Bassett@kent.ac.uk

"I'll try to respond to your question later, though my instinctive answer is that there's no real difference between the two statements. One way of approaching the whole thing is via Bayes' theorem (though this isn't a "bayesian" solution).

One quick point, though. The problem is *not* the same as prisoners' dilemma; that's a quite different two-person non-zero-sum game theory problem: see

http://en.wikipedia.org/wiki/Prisoners_dilemma

"

From Rebecca Pillinger R.J.Pillinger@bristol.ac.uk

"It depends exactly what the 'rules' are. The only rule mentioned in this description is that the king will choose 2 prisoners to be pardoned and the remaining prisoner will be executed. In order for the situation to be equivalent to the Monty Hall problem, an additional rule is required: the king tells the warder that the warder may pick any of the three prisoners, and the king will tell the warder the name of one of the other (non-selected) prisoners who will be pardoned. It is also necessary that the king decides who will be pardoned and who executed before the warder tells the king which prisoner he picks.

If the king tells the warder that he may pick any of the three prisoners, and then the king will tell the warder the fate of one of the other prisoners, but he does not say he will definitely give the name of a prisoner who will be pardoned, so that he has the option of naming the prisoner who will be executed, then even if he doesn't take this option and gives the name of a prisoner who will be pardoned, the situation is no longer equivalent to the Monty Hall problem.

We can see this by considering all the possibilities: there are initially 3 equally likely situations, Prisoner A will be executed, Prisoner B will be executed, or Prisoner C will be executed. If the warder then picks Prisoner A, in each of these situations the king has 2 options (which we'll assume he chooses with equal probability): if Prisoner A will be executed, he can say either that Prisoner B will be pardoned or that Prisoner C will be pardoned, each with 1/6 unconditional probability (e.g. P(Prisoner A executed (=1/3)) x P(king chooses to say Prisoner B pardoned (=1/2))). If Prisoner B will be executed, he can say that Prisoner B will be executed or that Prisoner C will be pardoned, again with 1/6 probability each. And if Prisoner C will be executed, he can say that Prisoner B will be pardoned or that Prisoner C will be executed, also with 1/6 probability each. So if the king says that Prisoner B will be pardoned, then it could be that Prisoner A will be executed or that Prisoner C will be executed and these are equally likely.

Thus Prisoner A can conclude he has a 50% chance of execution. (If the king had promised to name a prisoner who would be pardoned and denied himself the option of naming a prisoner who would be executed, then he would still have two options with 1/6 unconditional probability when Prisoner A is to be executed, but when Prisoner B is to be executed he must say Prisoner C will be pardoned, so this option has unconditional probability P(B executed (=1/3)) x P(king says C pardoned (=1)) = 1/3, and similarly when Prisoner C is to be executed. So if the king says Prisoner B is to be pardoned, it's twice as likely that Prisoner C is to be executed as it is that Prisoner A is to be executed- i.e. we do indeed have the Monty Hall problem).

The assumption that the king makes his choice before the warder tells him which prisoner he picks is really what you are asking about. (Assuming that the king promises he will name a different prisoner who will be pardoned), if the king chooses which prisoner will be executed and then the warder picks a prisoner, the situation is the Monty Hall problem. But if the king hasn't made up his mind, and when the warder picks a prisoner, names one of the others on the spur of the moment who he guarantees to pardon, then the king must go on to choose which prisoner to execute, and he has now left himself a choice of 2. Assuming that the warder's choice of prisoner does not influence the king- that he is not more likely to pardon that prisoner to please the warder nor to execute the prisoner to spite the warder- then the king is equally likely to choose between the two prisoners and Prisoner A has a 50% chance of execution. (Compare to the Monty Hall scenario: the equivalent would be if the host said that the guest could choose a door, and after the choice was made, the host would open a different door, and then put a goat behind one of the remaining 2 doors and a car behind the other). Prisoner A receives exactly the same information as if the king had already made his decision before the warder picked Prisoner A, but what the information means is different because the situation is different.

In the first case, the king's decision was completely independent of the warder's pick- it happened before the pick was even made. In the second case, the king's decision is not independent of the warder's pick- what happens is effectively that the king says he will pick one of the prisoners to be spared, but it is not allowed to be the prisoner the warder picks. So when the warder has picked Prisoner A, and before the king speaks, Prisoner A has 0 chance of being this first prisoner spared, and Prisoners B and C each have 50% chance. The king then picks one of B and C (say B), and tells the warder which one. C and A then each have a 50% chance of being the second prisoner picked to be spared.

So after the warder makes his pick and before the king speaks, the total chances of being pardoned are like this: for A, 0 of being first picked and 1/2 of being second (=1/2), for B, 1/2 of being the first picked, and 1/2 x 1/2 of not being the first picked but being the second picked (=3/4), and the same for C. So A has 1/2 chance of being executed and B and C each have 1/4. In other words, if the king makes his choice after the warder picks a prisoner, and the warder wants to help Prisoner A, he should pick one of the others!

This all assumes that the king has been making his statements because the king and the warder have an agreement that the warder will pick one of the prisoners and the king will tell the warder the fate of one of the other prisoners (in some cases we assumed he had to pick a prisoner who would be pardoned and in others he was allowed a completely free choice). But from the phrasing of your question, it's possible that the warder doesn't pick a prisoner, the king just spontaneously decides to tell him that Prisoner B will be pardoned, and either 1) he's made up his mind which of the other 2 will be executed but isn't saying or 2) he hasn't yet decided whether A or C will be executed.

In the case of 2), Prisoner A still has a 50% chance of execution because the king again has left himself 2 equally likely options to choose from. In the case of 1), we have to consider just what's motivating his statement. If he decided he was going to choose one of the prisoners at random and to tell the warder that prisoner's fate (whether the picked prisoner was to be pardoned or executed), then we have 9 equally likely scenarios:

the king could have chosen A to be executed and told the warder A would be executed, or that B would be pardoned, or that C would be pardoned; or he could have chosen B to be executed and told the warder A would be pardoned, or that B would be executed, or that C would be pardoned, or he could have chosen C to be executed and told the warder A would be pardoned, or that B would be pardoned, or that C would be executed. We know that actually the king told the warder B would be pardoned. This gives us 2 of the equally likely cases- A will be executed or C will be executed. So again A has a 50% chance of execution.

Alternatively, the king could have decided before speaking that he was going to reveal the name of one of the prisoners who would be pardoned. Before he speaks that rules out three of those 9 options, giving the king 6 equally likely possibilities (from our point of view): 1) A executed, king names B as pardoned, 2) A executed, king names C as pardoned, 3) B executed, king names A as pardoned, 4) B executed, king names C as pardoned, 5) C executed, king names A as pardoned, 6) C executed, king names B as pardoned. When the king says B is pardoned, we are left with 2 of these 6 equally likely possibilities, 1) and 6). In 1) A is executed and in 6) A is pardoned. So again, A has a 50% chance of execution. Thus if the king's statement is not in response to the warder's having picked a prisoner but is spontaneous, then it turns out not to matter whether the king is picking a prisoner to reveal the fate of at random or whether he has decided to reveal the name of a pardoned prisoner- either way when he says Prisoner B has been pardoned, Prisoner A has a 50% chance of execution.

In the case when the king does offer to reveal the name of a pardoned prisoner other than the prisoner which the warder picks, the non-50% probabilities come about either because the warder's pick limits the king's decision of who to pardon (in the situation when the king makes his decision after the warder picks a prisoner) or because the warder's pick limits the statements the king is allowed to make (in the situation where the king makes his decision first and then the warder picks a prisoner).

In this second case, the unequal probabilities come about because the limits on what the king can say alter the information conveyed by his statement. Imagine that after the warder picks A the king flips a coin. In the case where the king is allowed to name a pardoned or an executed prisoner, then the king will choose B if the coin comes up heads and C if the coin comes up tails. In the case where he must name a pardoned prisoner, he will still flip the coin, but he will only use it to choose a prisoner if A is to be executed- if B or C is to be executed he must ignore the coin and choose the other one. So in both cases there are 6 equally likely situations- 3 possibilities for which prisoner is to be executed and in each of those 2 possibilities for which way the coin comes up. In each of those 6 possibilities the king's choice of what to say is determined- either by which way the coin came up or by the rules only giving him one option (when he must name a pardoned prisoner and B or C is going to be executed). If the king does not have to name a pardoned prisoner, then his naming B as pardoned tells us that the coin must have come up heads (since in this case his choice of which prisoner to name is determined by the coin toss) and that it was either A executed - heads or C executed - heads (it cannot have been B executed - heads)- so there is a 50% chance of A executed. If the king does have to name a pardoned prisoner, and names B, it could be that A is to be executed and the coin came up heads, or it could be that C is to be executed and the coin could have come up heads or tails since if C is to be executed the king must choose B and the decision is not based on the coin toss- in this case just one of the three possibilities involves A being executed. So depending on what the rules were, the king's statement puts us in one of 2 equally likely possibilities, one of which involves A being executed, or one of 3 equally likely possibilities, again one of which involves A being executed. Depending on the rules therefore the king's statement gives us different information.

The same reasoning allows us to compare the information conveyed if the king must named a pardoned prisoner but the warder does not make a pick first, and if the king must name a pardoned prisoner and the warder does pick a prisoner whom the king is not allowed to name. In the first case, the king's coin toss again always decides who he will name- if A is to be executed, the king will name B if the coin comes up heads and C if the coin comes up tails; if B is to be executed the king will name C if the coin comes up heads and A if the coin comes up tails; and if C is to be executed the king will name A if the coin comes up heads and B if the coin comes up tails; in the second case as described above the king uses the coin toss to decide only if the prisoner the warder picks is the prisoner who will be executed (if the warder picks a pardoned prisoner the king is forced to name a particular one of the other two by the rules). So in the first case if the king names B then it could be A executed and heads or C executed and tails - 2 equally likely possibilities of which 1 involves A being executed, and in the second case if the king names B then as we saw above there are 3 equally likely possibilities of which one involves A being executed. Again, different information is given by the king's statement according to whether the warder is allowed to pick a prisoner or not.

Thanks for an interesting question- that certainly got me thinking!"

And a further email from Rebecca…

"I did get a bit carried away, but it was an interesting question!

I had another thought after I wrote that: when you ask whether there is a difference in probability depending on whether the king has made his mind up completely or only decided so far that B will be pardoned, you say 'prisoner A will receive the same information regardless.' In fact, knowledge of the rules of the game is in itself information. If prisoner A does literally get the same information in both cases- if the warder simply says 'The king has said he will pardon 2 of you and execute the third, and he's also said B will be pardoned'- then from prisoner A's point of view in either case the best estimate of the chance of execution is 50%. But if the warder also tells A in the first case that the king had already made up his mind as to which prisoner to execute, and that the king had told the warder he could pick a prisoner and the king would name a different prisoner who would be pardoned, then A has the information to work out the chance of execution is 1/3. In the second case, the warder would instead say that the king had decided that B will be pardoned, but that he had not yet decided which out of A and C to execute, and with this different information A could conclude the chance of execution was 1/2."

From Iain Third iain_third@hotmail.com

"That sounds similar to the variant of the Monty Hall problem where the game show host has forgotten which door the car is behind, and having decided to bluff it and randomly opens a door, luckily opening a door with a goat behind it.

In that variant the probability of having the car is 0.5 because you no longer have the extra information from the knowledge of the host, so switching is no longer an advantage."

My reply to Iain

"But only if the contestant realises that the host has forgotten and is bluffing. If he doesn't realise then as far as he is concerned it is still better to switch."

Further response from Iain Third iain_third@hotmail.com

"If he doesn't realise that the gameshow host is bluffing then he may think it is better to switch, in actual fact it's 50/50, so switching does no harm anyway, but whether he knows it's a bluff or not the advantage of switching is lost.

When you pick a door the two remaining doors have a 0.66 chance of having the car at the outset. The hosts knowledge means he opens a door with a goat behind it leaving the remaining door as 0.66 probability of having a car. If the host is bluffing their are two scenarios. He opens a door on a goat or he opens a door on a car by mistake. The odds on him opening a car are 0.33, and a goat are 0.66. That 0.66 splits two ways, the other goat is the door you picked of the other goat is the other door he could have picked. If we assume he has not picked the car by mistake then we are 100% certain he has picked a goat, therefore the odds on your door being a car are half of that, so 0.5."

Mike Weale (KCL) michael.weale@kcl.ac.uk

"It all depends on the underlying set of rules that the King is using to pick prisonners to pardon, and also on the reasons for why he tells the Warden about B. Let's start with the case of an arbitrary King who picks prisoners to pardon at random, and who also tells the Warden about his picks regardless of which prisoner they are. In this case, P(A to be pardoned | B has been pardoned)=0.5 in both your scenarios - knowledge of B really does alter your posterior probabilities here.

Now let's assume the King still picks arbitrarily, *but* he knows that the Warden is friends with A and so only chooses to give him information about other prisoners. In this case, P(A to be pardoned | King tells Warden about B)=1/3 in your first scenario - because the King can always chose someone else to tell the Warden about. However, in your second case, P(A to be pardoned | B has been pardoned)=05 again, because the fact that the Warden gets told anything is dependent on A not being chosen first.

All of which is of little consolation to A. He might find the intellectual stimulation a welcome distraction from his predicament, but I doubt it."

3a. My Second Query to ALLSTAT

"Thank you very much for all the responses I have received so far. They have been fascinating and include one that must qualify for the longest reply ever on ALLSTAT. What I will do with these replies is create a PDF file with all of them and put a link to that on my website. However, before I do that, I want to clarify my query a little as your responses showed I hadn't fully specified the rules of the game.

The reason why I put the query out is that I use the Monty Hall problem as an ice-breaker in my training courses. It is a great way of getting people to realise that statistical thinking is essential since normal human intuition fails in this problem. However, I do like to then make the Monty Hall problem real by translating it into a context that people are likely to experience in real-life. My example wasn't the prisoners & king which I used in my previous email but a competitive tender involving 3 companies A, B, C who bidding for a contract with client X (could equally be 3 people applying for a job).

The rules of this game are. You are working for company A and to begin with all 3 companies tendering can be regarded as equally likely to win. You decide you want to change the odds by bribing someone who works for X (call him P) to find out what the person (call her Q) in X who will be making the decision, is currently thinking about B and C. We will suppose that P and Q are quite friendly with each other and are having a chat when P casually says "I know a bit about company A since I've worked them in the past but I don't know much about B & C. What are your impressions so far of B & C?". Let's now imagine two responses that Q could give:

1. "Definitely not choosing B! Can't tell you yet which of A & C I will be choosing but I have made my mind up."
2. "Definitely not choosing B! However, I haven't decided whether C is better than A."

Both statements contain different information but it is now up to P to convey the information back to you. In this scenario, it seems to me that if you are the salesman for A, if you hear P relay statement 1 correctly then you should conclude that the Monty Hall problem applies i.e. you are playing a stick strategy and you now know that C has a 2/3 chance of winning. Therefore you have wasted your money bribing P and you should now resign from A and go and work for C. But if you hear statement 2 from P, does the Monty Hall rationale still apply? Finally what if P provides you with incomplete information and only says "It's not going to be B" and leaves out the second sentence from Q? What conclusions can you as the salesman for A make?"

3b. Responses to My Second Query

From Braunholtz, David A. d.braunholtz@abdn.ac.uk

"Surely there is some extra info required, about how Q is making her decision, and how she decides what to say. Monty Hall type reasoning applies (of course) only if the choice is random, AND if it is known she will name one of B , C as not having been selected. In the case she answers 2. she presumably is first rejecting one of the 3, then will proceed to reject another. The info that A and C have not been rejected at stage (i) leads to a 50:50 probability for A and C. The difference is that she will not always be ABLE to answer 2. - viz if the first random rejection is of A she would have to say that A has been rejected. So knowledge that A has not been rejected in the first round is imparted etc"

My response to David

"I'm not sure if I agree with your reasoning here. The Monty Hall show requires the contestant to choose a door (A say) and then the host is restricted to giving information about B or C. In my tender, you as the salesman have already chosen your first door by working for A and I have then constructed the problem so that you ask P to find out some information from Q about B or C but NOT about A specifically. The whole point about Monty hall is that no information about A is explicitly given but you have to draw inferences about the probability of A winning from the indirect info received."

Further response from Braunholtz, David A. d.braunholtz@abdn.ac.uk

"The MH scenario requires

a) random allocation of winner to A / B / C

b) that player knows that MH will ALWAYS (truthfully) nominate one of A or B as a loser

This scenario leads to the 1/3 : 2/3 posterior probabilities for A : (whichever of B, C not nominated as loser) winning

The alternative scenario 2. you set up below does not state what we can expect Q to say under different circumstances.

We might assume that (at least as far as the player A is concerned) the choice is made randomly and in two stages - first choose one loser, then the other. And then consider what Q might say after the first stage. We could suppose that if B or C have been chosen as a loser, then she will indicate this. However, what do we assume she will say if A has been chosen as the first loser ? Presumably she has to be honest (another assumption) - and so could not indicate either A or B as the loser. It would then be clear to the player that A is a loser. So (with these assumptions) the statement made by Q in scenario 2 always completely identifies the first round loser, leaving a probability of 0.5 for each of the other two being the eventual winner.

It is easy to omit the requirement b) from consideration, but without it you don't know where you are (the problem becomes indeterminate)"

Meyners, Michael Michael.Meyners@rdls.nestle.com

"Focussing on the statistical part, I believe that the additional information (except for "definitely not B") cannot make a difference.

The only reliable information you get is that you should not choose B.

Every information you seem to get from the second statement relies on psychological interpretation. Indeed, if you know P very well, you might know what either statement might mean or at least indicate, but you will have to translate this into probabilities and hence into your statistical model in order to move away from 1/3 vs. 2/3. As long as we (or the salesman) do not have reliable and quantitative information on this psychological part, the information is useless. In other words: The conclusion for A is always the same, with (or without) any of the additional statements."